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Some Transformations of Monotone Sequences*

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Abstract. In this paper, we discuss a class of methods for summing sequences which are generalizations of a method due to Salzer. The methods are not regular, and in contrast to the classical regular methods, seem to work best on sequences which are monotone. In our main theorem, we determine a class of convergent sequences for which the methods yield sequences which converge to the same sum.

1. Introduction. In this paper, we discuss a class of transformations which are useful for summing certain monotone sequences.

In what follows, let k, m, n be integers, $k \geqq 0, m, n \geqq 1$, let $\{S_n\}$ be a sequence of complex numbers and λ be a complex number. The transformation $\mathfrak{U}_m$: $\{S_n\} \to \{S_n^*\}$ is defined by

$$S_m^* = \mathfrak{U}_m(\{S_n\}) = \mathfrak{J}_m(\{S_n\})/\mathfrak{J}_m(\{1\}), \tag{1}$$

where

$$\mathfrak{J}_m(\{S_n\}) \equiv \mathfrak{J}_m(k, \lambda, \{S_n\}) = \frac{1}{m!} \sum_{r=1}^{m} (-)^{r+1}(\lambda + r)^{k+m}\binom{m}{r} S_r. \tag{2}$$

$\mathfrak{U}_m$, for the case where $k = 0$, $\lambda = -N$, was discussed by Salzer [1], [2], [3]. He was interested in using $\mathfrak{U}_m$ as a summation process for converting the slowly convergent (or divergent) sequence $\{S_n\}$ into a more rapidly convergent (or convergent) sequence $\{S_n^*\}$. Although Salzer provided no convergence criteria, he did furnish a number of practical examples where $S_n^* \to \alpha$ when $S_n \to \alpha$. This is not, however, true in general, as is shown in Section 2. In Section 3, we describe a class of sequences for which S_n^* is convergent, and we close with an example.

2. The Regularity of $\mathfrak{U}_m$. $\mathfrak{U}_m$ is not regular. To show this, we require a

LEMMA.

$$\begin{aligned} \mathfrak{J}_m(\{1\}) &= \frac{1}{2\pi i} \int_{\Gamma_m} \frac{(\lambda + z)^{k+m}}{(-z)_{m+1}}\, dz \\ &= \frac{\lambda^{k+m}}{m!} + \frac{(-)^{m+1}}{m!} (k + 1)_m B_k^{(-m)}(\lambda) \\ &= A(-)^{m+1} m^{2k}[1 + O(m^{-1})], \qquad m \to \infty,\ A \neq 0, \end{aligned} \tag{3}$$

where Γ_m is a simple closed curve encircling the integers $1, 2, \cdots, m$, $\operatorname{Re} z > 0, z \in \Gamma_m$, *and* $(\mu)_k = \mu(\mu + 1) \cdots (\mu + k - 1)$.

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Proof. Evaluate the integral above by residues. This gives (2) with $S_r \equiv 1$. Now use the formula

$$(\lambda + r)^{k+m} = \sum_{s=0}^{k+m} \binom{k+m}{s} (-)^s (-r)_s B_{k+m-s}^{(-s)}(\lambda), \tag{4}$$

(Nörlund [4, p. 150]), substitute this in (2) and interchange the order of summation. The lemma follows immediately if we use the fact that the Bernoulli polynomial $B_k^{(-m)}(\lambda)$ is a polynomial in m of degree k.

To show $\mathcal{U}_m$ is not regular, let $\{S_n\}$ be the null sequence

$$S_n = (-)^{n+1}/(n + \lambda), \tag{5}$$

and suppose $\lambda > 0$. Then

$$\begin{aligned} |\mathfrak{T}_m(\{S_n\})| &= \frac{1}{m!} \sum_{r=1}^{m} (\lambda + r)^{k+m-1} \binom{m}{r} \\ &\geqq \left(\lambda + \frac{m}{2}\right)^{k+m-1} \Big/ \Gamma\left(\frac{m}{2} + 1\right)^2 \\ &> Ce^m m^k, \quad \text{for } m \text{ even and some } C > 0, \end{aligned} \tag{6}$$

so, by using (3), we see that

$$S_m^* > De^m m^{-k}, \qquad m \text{ even and for some } D > 0, \tag{7}$$

and S_m^* does not converge.

3. A Class of Sequences Summed by $\mathcal{U}_m$. A distinguishing characteristic of the transformation $\mathcal{U}_m$ is, generally speaking, that it sums best (in the sense that if $S_n \to \alpha$, then $S_n^* \to \alpha$ more rapidly) those sequences which are monotone, and is less effective on summing sequences which are not monotone. Exactly the opposite is true of most of the classical summation procedures, such as the Cesaro summability method, which work best on those sequences the successive differences of whose members alternate in sign.

In the following theorem, we explore this interesting feature of $\mathcal{U}_m$ by determining a class of sequences for which $S_m^* \to \alpha$ if $S_m \to \alpha$.

Let in what follows

$$S_n = \alpha + R_n, \qquad S_n^* = \alpha + R_n^*. \tag{8}$$

THEOREM. *Let there exist a function $R(z)$ analytic for* Re $z \geqq p$ *for some* p, $0 < p < 1$, *such that*

$$R(z) = O(|z|^{\mu}) \quad \textit{for some} \quad \mu < -k, \tag{9}$$

as $|z| \to \infty$, Re $z \geqq p$. *Further, let*

$$R(n) = R_n, \qquad n \geqq 1. \tag{10}$$

Then

$$\mathcal{U}_m(\{S_n\}) = \alpha + o(m^{-2k}), \qquad m \to \infty. \tag{11}$$

Proof. Under the given conditions, we have

$$\alpha - \mathfrak{U}_m(\{S_n\}) = R_m^* = \frac{\mathfrak{J}_m(\{1\})^{-1}}{2\pi i} \int_{\Gamma_m} \frac{(\lambda + z)^{k+m}}{(-z)_{m+1}} R(z)\, dz$$
$$(12) \qquad = \frac{\mathfrak{J}_m(\{1\})^{-1}}{2\pi i} \int_{p-i\infty}^{p+i\infty} \frac{(\lambda + z)^{k+m}}{(-z)_{m+1}} R(z)\, dz,$$

so

$$(13) \qquad |R_m^*| \leqq Cm^{-2k} \int_{-\infty}^{\infty} |p + iu|^{\mu-1}\, |p + \lambda + iu|^k \prod_{j=1}^{m} \frac{|p + \lambda + iu|}{|p - j + iu|}\, du.$$

We can choose constants A and $\epsilon > 0$ such that

$$(14) \qquad |p + iu|^{\mu-1}\, |p + \lambda + iu|^k \leqq A(|u| + \epsilon)^{\mu+k-1}, \qquad -\infty < u < \infty.$$

Then, we have

$$(15) \qquad |R_m^*| \leqq C' m^{-2k} \int_0^{\infty} (u + \epsilon)^{\mu+k-1} \prod_{j=1}^{m} \frac{[(\lambda + p)^2 + u^2]^{1/2}}{[(j - p)^2 + u^2]^{1/2}}\, du.$$

Now, we may write

$$(16) \qquad \frac{(\lambda + p)^2 + u^2}{(j - p)^2 + u^2} = 1 + \frac{(\lambda + p)^2 - (j - p)^2}{(j - p)^2 + u^2},$$

so it is clear we may choose j_0, $1 \leqq j_0 \leqq m$ so that the left-hand side above is monotone increasing in u for $j > j_0$ and monotone decreasing for $j \leqq j_0$ (in fact, $j_0 =$ sup {integral part $(\lambda + 2p)$, 1}).

We have

$$(17) \qquad |R_m^*| \leqq C'' m^{-2k} \left\{ \int_0^{m^{1/2}} + \int_{m^{1/2}}^{\infty} (u + \epsilon)^{\mu+k-1} \prod_{j=j_0+1}^{m} \frac{[(\lambda + p)^2 + u^2]^{1/2}}{[(j - p)^2 + u^2]^{1/2}}\, du \right\}$$

$$(18) \qquad \leqq C'' m^{-2k} \left\{ K \prod_{j=j_0+1}^{m} \frac{[(\lambda + p)^2 + m]^{1/2}}{[(j - p)^2 + m]^{1/2}} + \int_{m^{1/2}}^{\infty} (u + \epsilon)^{\mu+k-1}\, du \right\}$$

$$(19) \qquad \leqq C'' m^{-2k} \left\{ K \frac{[(\lambda + p)^2 + m]^{m/2}}{(j_0 + 1 - p)_{m-j_0}} + \frac{(m^{1/2} + \epsilon)^{\mu+k}}{(\mu + k)} \right\}$$

$$(20) \qquad = o(1) m^{-2k}.$$

Equation (20) follows from (19) when Stirling's formula is used. This completes the proof.

As an example, let

$$(21) \qquad S_n = \sum_{k=1}^{n} [k(\omega + k)]^{-1}, \qquad \omega \neq -1, -2, -3, \cdots,$$

so that

$$(22) \qquad \alpha = (\psi(\omega) + \gamma + \omega^{-1})\omega^{-1},$$

$$(23) \qquad R_n = (\psi(n + 1) - \psi(\omega + n + 1))\omega^{-1}.$$

We can let

$$(24) \qquad R(z) = (\psi(z + 1) - \psi(\omega + z + 1))\omega^{-1} = -z^{-1} + O(z^{-2}),$$

arg $z < \pi$. Thus, the theorem shows that for $k = 0$, $S_m^* \to \alpha$. Unfortunately, the theorem is far too conservative. It provides an estimate of $o(1)$ for R_m^*, while numerical evidence shows that R_m^* goes to zero much more rapidly than R_m, which is $O(m^{-1})$ as $m \to \infty$.

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